Truck tyre rolling resistance under dynamic vertical load

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Abstract

This report deals with tyre modelling to predict rolling resistance of truck tyres under dynamic vertical load. Several models are applied and modified to perform the desired calculations. The model predictions are compared with the only available experimental data on rolling resistance under dynamic vertical load at the time of writing [18]. The analysis is extended into a larger frequency range so that the different models considered can be assessed, and recommendations are made as to what model is best suited for the purpose of this project.
Nomenclature

Flexible ring model (Section 2.1)

\[ \alpha \] transfer function matrix obtained by the modal synthesis method
\[ \beta \] rotation angle of the tread band cross-section
\[ \theta_r \] mean angular displacement of the tread band with respect to the wheel hub
\[ \rho A \] mass density of the ring
\[ \sigma_0^{\theta} \] initial stress in the ring
\[ \omega \] frequency
\[ \Omega \] average angular velocity of the wheel-tyre system
\[ A \] area of the cross-section of the ring
\[ b \] width of the ring
\[ c_t \] damping coefficient of simple spring and dashpot tyre model
\[ (c_v, c_w) \] damping coefficients per unit length of the sidewalls in the tangential and radial directions
\[ EI \] bending stiffness of the ring
\[ f_n = \{ \xi_n \ \eta_n \} \] generalised vector of external forces
\[ f_x, f_x^\ast \] external longitudinal force acting at the wheel axle
\[ f_z, f_z^\ast \] external vertical force acting at the wheel axle
\[ G_n \] modal damping-gyroscopic matrix
\[ H \] transfer function matrix between the contact patch and the wheel for the free tyre-wheel system
\[ h_0 \] tread rubber thickness
\[ I_r \] moment of inertia of the wheel
\[ K_n \] modal stiffness matrix
\[ (k_{EI}, k_{G1}) \] stiffness per unit length of the tread rubber in the radial and tangential directions
\[ k_t \] spring stiffness of simple spring and dashpot tyre model
\[ K_{T,1}(\omega) \] vertical dynamic tyre stiffness for cleat tests
\[ K_{T,2}(\omega) \] vertical dynamic tyre stiffness for axle vibrations
\[ (k_v, k_w) \] stiffness per unit length of the sidewalls in the tangential and radial directions
\[ m \] mass of the wheel body
\[ n \] vibration mode number
\[ M_n \] modal mass matrix
\[ p_0 \] inflation pressure of the tyre
$(p_v, p_w)$

external point force acting on the ring at the contact point in the tangential and radial directions

$q_\beta$

external moment per unit length acting on the ring in the contact patch

$(q_v, q_w)$

external tangential and radial forces per unit length acting on the ring in the contact patch

$(r, \theta)$

polar coordinates of a point on the tyre in the rotating coordinate system

$(r, \phi)$

polar coordinates of a point on the tyre in the non-rotating coordinate system

$R$

mean radius of the ring

$s$

Laplace variable

$t$

external torque torque applied at the wheel axle

$u_n = \begin{\{a_n b_n\}}$

generalised vector of modal displacements

$V_x$

forward speed of the vehicle

$(v, w)$

tangential and radial displacements of a point on the inextensible ring

$(x, z)$

Cartesian coordinates of a point on the tyre in the non-rotating coordinate system

$(x^*, z^*)$

Cartesian coordinates of a point on the tyre in the rotating coordinate system

**Radially distributed hysteretic springs (Section 2.2)**

$\delta_0$

static deflection of the tyre (also maximum deflection seen by each spring element)

$\Delta \theta$

angular dimension over which the continuous elasticity of the tyre (in the radial direction) is lumped

$\Delta W_t$

energy or work consumed by each spring element per tyre revolution

$\omega$

frequency

$a$

hysteresis coefficient per unit angle around the tyre

$c_t$

damping coefficient of simple spring and dashpot tyre model

$F_x$

rolling resistance derived from the energy consumed during the rolling cycle

$F_t$

dynamic vertical load applied at the axle

$k_{Et}$

stiffness per unit length of the tread rubber in the radial direction

$k_t$

spring stiffness of simple spring and dashpot tyre model

$k_w$

stiffness per unit length of the sidewalls in the radial direction

$K$

radial spring stiffness per unit angle around the tyre

$(m_s, m_t)$

sprung and unsprung mass of quarter-vehicle model

$R_e$

effective rolling radius of the tyre

$R_0$

radius of the undistorted tyre

$W_t$

total energy consumed by the tyre per revolution

$y$

road profile displacement input to the tyre

$y_t$

vertical displacement of the unsprung mass
Pacejka’s transient tyre model (Section 2.3)

\( \varepsilon \) effective rolling radius gradient \(-\frac{\partial r_e}{\partial d_t}\)

\( \zeta \) speed dependent damping ratio

\( \eta \) effective rolling radius gradient \(-\frac{\partial r_e}{\partial \rho}\)

\( \kappa \) longitudinal wheel slip ratio

\( \kappa' \) transient longitudinal tyre slip

\( \nu \) non-dimensional frequency

\( \rho \) tyre radial (vertical) deflection

\( \sigma_\kappa \) relaxation length associated with tyre longitudinal slip

\( \omega \) frequency

\( \omega_{\Omega_0} \) natural frequency of the tyre-wheel rotation with respect to the contact patch

\( \Omega \) wheel speed of revolution

\( A_r \) rolling resistance coefficient

\( C_{F\kappa} \) longitudinal slip stiffness

\( C_{Fx} \) longitudinal tyre stiffness

\( C_{Fz} \) tyre radial (vertical) stiffness

\( d_t \) tyre tread thickness

\( F_\kappa \) tangential slip force

\( F_r \) rolling resistance force

\( F_x \) longitudinal tyre force

\( F_z \) vertical (normal) tyre force

\( I_w \) wheel polar moment of inertia

\( M_y \) rolling resistance moment

\( r \) tyre loaded radius

\( r_c \) radius of unloaded tyre carcass (belt)

\( r_e \) effective rolling radius of the tyre

\( r_f \) free undeformed tyre radius

\( s \) Laplace variable, travelled distance

\( t \) time

\( u \) longitudinal deflection of the tyre

\( V \) forward speed (assumed to be constant)

\( V_{sx} \) longitudinal slip velocity

\( w \) vertical road (effective) profile

\( (x, z) \) longitudinal and vertical axle displacements

\( X \) longitudinal horizontal tyre force
Brush model and contact patch length (Section 2.4)

\( \zeta_{cx} \) theoretical longitudinal slip
\( \kappa_{c} \) practical longitudinal slip
\( \Omega \) wheel speed of revolution
\( a \) half the contact patch length
\( F_{cx} \) longitudinal tyre force in the contact patch
\( F_z \) tyre vertical load
\( k_{cp} \) longitudinal tread stiffness per unit length
\( q_{a1}, q_{a2} \) coefficients of the second order polynomial expressing half the contact patch length as a function of the square root of the vertical load
\( R \) free undeformed tyre radius
\( r_e \) effective rolling radius of the tyre
\( s \) position in the contact patch relative to the centre of the contact patch
\( u \) longitudinal deformation of a tread element
\( V_{c,sx} \) longitudinal slip velocity
\( V_{cr} \) rolling velocity in the contact patch
\( V_{cx} \) forward velocity in the contact patch
\( x \) travelled distance

Flexible ring model with tyre-road interface (Section 2.5)

\( \theta_i \) discrete angular position along the ring circumference / in the contact patch
\( a \) half the contact patch length
\( f_n = \{ \xi_n \eta_n \} \) generalised vector of external forces
\( F_x \) longitudinal force in the contact patch
\( G_n \) modal damping-gyroscopic matrix
\( (k_{e1}, k_{e1}) \) stiffness per unit length of the tread rubber in the radial and tangential directions
\( K_n \) modal stiffness matrix
\( M_n \) modal mass matrix
\( n_e \) number of contact points in the tyre-road interface
\( n_m \) number of vibration modes
\( (q_{v,i}, q_{w,i}) \) tangential and radial forces per unit length of a discrete point in the contact patch
\( (q_{x,i}, q_{z,i}) \) longitudinal and vertical forces per unit length of a discrete point in the contact patch
\( R \) mean radius of the ring
\( u_n = \{ a_n b_n \} \) generalised vector of modal displacements
\( (v, w) \) tangential and radial displacements of a point on the inextensible ring
\( (v_i, w_i) \) tangential and radial displacements of a discrete point on the inextensible ring
\( (x_i, z_i) \) horizontal and vertical displacements of a discrete point in the contact patch
Chapter 1

Introduction

A pneumatic tyre is an essential part of a vehicle, as it is the only part of the vehicle which is in contact with the road: all the interaction forces between the road and the vehicle are transmitted at the tyre through a very small contact area, generally called the ‘contact patch’ or ‘footprint’. Therefore, a good understanding of the behaviour of tyres is very important for vehicle dynamics.

1.1 Structure of the tyre

The behaviour of the tyre is governed by its complex structure (see Figure 1.1). It is composed of high modulus flexible filaments such as textile, metal or glass, embedded in and bonded to a low modulus matrix (rubber or rubber-like polymer) [6].

A tyre has three main components:

- Tyre carcass: flexible filaments of high modulus cord, natural textile, synthetic polymer, glass fibre or fine hard-drawn steel. There are usually multi-filaments layers or plies. In a layer, the cords must be in one direction, but they are frequently connected by light-weight wefts for manufacturing processing.
Tyre casing: the layers of the high modulus cord or filaments are turned around bead coils made of a number of high tensile, hard drawn steel wire, located at the inner edge of the tyre sidewalls. The steel bead coils may have a canvas wrapping. In a number of tyre designs, security of the attachment of tyre to the rim is increased by an interference fit between bead and rim diameters.

Tyre tread: it is the only part of the tyre in contact with road. It protects the casing and provides frictional contact to transmit driving, braking and cornering forces. The only type of material successfully used so far has been rubber (natural or synthetic) or rubber-like material. It is reinforced by suitable ingredients such as carbon black to obtain the required abrasion resistance.

There are two main types of tyres:

- Cross-bias tyre: it is generally composed of 4 layers of parallel cord filaments, the direction of the cords in a layer being at angle (crown angle, between 28 and 38 degrees) to the equator of the tyre. To increase the casing strength and protection, one or two layers, called ‘breakers’, of cords are sometimes incorporated substantially parallel to the cords in the other plies but extending only approximately the width of the tyre tread.

- Radial ply rigid breaker tyre: the filaments are disposed in an approximate radial direction, giving a crown angle of approximately 90 degrees. There is also a breaker or belt of several plies of cord fitted on top of the casing under the tread and laid at various crown angles. The radial tyre is the most widely used tyre for cars and trucks, see Figure 1.1.

1.2 Existing tyre models

The selection of a suitable tyre model is a difficult task, since a wide range of formulations exists. A ready comparison of model capabilities and limitations from which relative performance can be assessed is not available. Furthermore, overly detailed and sophisticated models lead to a high penalty in terms of setup time and computing cost [1].

Very few studies have been concerned with vibration transmission properties of tyres including both experimental and theoretical data. Among these, Gong et al. [4, 5] developed a theoretical model of the tyre alone, without including the tyre-road interface. The system modelled was a tyre-wheel system represented by a ring model, which lead to the calculation of time-dependent modes. The model was then applied to the study of rolling contact between a tyre and a flat road, and for obtaining the frequency response functions of a free tyre-wheel system. Finally, the vibration transmission of tyres under various boundary conditions was studied. Estimation of the parameters of the model using tyre dynamic tests allowed validation of the model.

Zegelaar [27] investigated the in-plane dynamics of tyres arising from different excitation sources (brake torque variations, road roughness, longitudinal and vertical axle motions and tyre non-uniformities). He used the flexible ring model developed by Gong [4] to study the behaviour of the tyre in detail and introduced a more compact rigid ring model for vehicle simulations. Other types of tyre models exist that can simulate the enveloping properties of tyres on
rough roads (see Figure 1.2): single-point contact model (spring and damper in parallel), roller contact model (rigid wheel with one spring, one damper and a single contact point), fixed footprint model (linearly distributed stiffness and damping in the contact area), radial spring model (circumferentially distributed independent linear spring elements), flexible ring model and finite elements models. Experimental modal analysis and theoretical results using the flexible ring model were presented in [26] for two boundary conditions, a free tyre and a tyre standing on the road.

![Various tyre models. From [27].](image)

Further work by Zegelaar et al. [28] concerned tyre vibrations and transmission properties between the tyre-road contact point and the wheel axle, employing the flexible ring model. A simplified model was developed, able to take into account the zeroth and first mode of the tyre, and a tyre-road interface based on the brush model.

Captain et al. [1] also presented four tyre models suitable for vehicle dynamics simulations (point-contact, rigid tread band, fixed footprint, adaptative footprint) and compared them in a six degree-of-freedom nonlinear simulation of a cargo truck crossing rough ground. They found that the differences between the tyre models were mainly at high frequencies and issued guidelines for the selection of an optimum tyre model. However, no comparison with experimental data was provided.

Popov et al. used the flexible ring model developed by Gong to model free rolling of tyres [14] and non-steady rolling of tyres [15] in order to investigate rolling resistance of truck tyres and compared their findings with experimental results [16, 17]. The parameters of the flexible ring model were obtained by experimental modal analysis of a tyre [12]. The power consumption of both the tyre and the suspension when travelling at constant speed over a rough road were quantified for a range of suspension damping values by incorporating the flexible ring model in a quarter-vehicle model [12].

### 1.3 Rolling resistance of pneumatic tyres

Rolling resistance is of primary importance for heavy vehicles as approximately one third of the energy consumed by a heavy vehicle engine is used to overcome its effects [20]. The source of rolling resistance is energy dissipation in both the tyres and the suspension of the vehicle. For a freely rolling tyre, the rolling resistance can be measured on large rotating drum with a smooth surface [19]. Rolling resistance arises from hysteretic losses in the sidewalls and tread band material, which experiences a deformation cycle every revolution of the tyre [16, 17]. There
is an additional small loss (approximately 10%) due to micro-slip in the contact patch between
the tyre and the test surface. Three major mechanisms exist by which the road roughness can
produce additional losses [16, 17] (see Figure 1.3):

- excitation of the vehicle by road roughness, leading to energy dissipation in the suspension
dampers and frictional losses due to vibration;
- hysteretic losses in the contact patch due to dynamic vertical deflection of the tyres, and
additional frictional losses in the contact patch due to micro-slip;
- hysteretic losses in the tyre material due to envelopment of the road roughness by the tyre,
which also cause further frictional losses in the contact patch due to micro-slip.

![Figure 1.3: Tyre rolling resistance. From [17].](image)

This energy dissipation results in a normal pressure distribution: higher in the forward
portion of the contact patch, where the tread elements are forced radially inwards; and lower
in the rearward portion of the contact patch, where the tread elements are forced radially out-
wards [21]. This leads to a forward shift of the centroid of the normal pressure distribution, at
a distance $e$ in Figure 1.3, where the horizontal and vertical forces applied at the wheel hub are
reacted. A moment balance yields the following relationship:

$$F_x h - F_z e = 0 \quad \Rightarrow \quad \frac{F_x}{F_z} = \frac{e}{h},$$

(1.1)

where $h$ is the loaded radius of the tyre. The ratio $e/h$ is by definition the rolling resistance of
the tyre.

Schuring wrote a very extensive review on rolling resistance of tyres in steady state
conditions and under constant vertical load [19]. He gave a general definition of ‘rolling loss’
which considered a ‘freely rolling’ tyre (i.e. with zero applied torque) as a particular case, split
between a ‘braked’ tyre and a ‘driven’ tyre. He defined ‘steady state’ as a rolling process where
all the variables (mechanical, kinematic and geometric) are fixed in time, whereas he referred
to an ‘equilibrium rolling loss’ as a description of steady-state losses that occur when the tyre
reaches a thermal equilibrium. He investigated the influence of operation conditions (speed, load,
inflation pressure, ambient temperature, tyre temperature, pavement texture, etc...) as well as tyre
design parameters (rubber type, tyre size, tread pattern, etc...) on rolling resistance. His findings
were mainly of an experimental nature although he did mention various tyre models (empirical,
thermal, viscoelastic and thermo-viscoelastic) to predict rolling resistance. These models
gave conflicting results, highlighting the difficulty in modelling rolling resistance accurately and
shading some doubt as to the validity of the predictions by the available theoretical models.

Segel and Lu [10, 21] developed a simple tyre model to evaluate rolling resistance for
vehicles travelling over a rough road and quantified the energy losses in both the tyre and the
suspension for a range of road surfaces and as a function of vehicle speed using a quarter-car
model. They modelled the viscoelastic behaviour of the tyre by distributed radial springs with
hysteretic characteristics and assumed that the energy loss per revolution of the tyre was the sum
of the energy lost in each hysteretic spring compressed during the rolling process. They found
that the energy lost in the tyre remains relatively constant with road surface at low speeds, but
the energy dissipated in the suspension increases dramatically when the road surface degrades or
when the vehicle speed increases. The impacts between tyre and road were also considered, and
were found to be significant only at high speeds or on very rough roads. Overall, Segel and Lu
found that, for an average road, the energy losses were approximately 10% higher than those on
an absolutely smooth surface at the same constant speed.

Klingbeil et al. [8, 9] developed a viscoelastic model for rolling resistance calculations
where rolling losses are associated with seven deformation mechanisms in the tyre structure.
These rolling losses are computed through harmonic analyses of all deformation cycles and the
application of a loss tangent factor to the maximum stored strain energy for each spectral com-
ponent. They used composite structures theory to compute the model stiffness parameters and
energy loss factors from the tread band design. The results showed that the main contributions
to rolling resistance were from bending of the tread band, compression of the tread and shearing
of the sidewalls. Comparison with experimental data showed a general good agreement with the
predicted trends but there were some significant discrepancies in the absolute values. Popov et
al. [17] used this method for truck tyres. However, they only considered the rolling resistance
component due to tread compression, which was found to amount to approximately 56% of the
measured value of rolling resistance.

between a tyre and a flat road. They used the flexible ring model with an auxiliary elastic foun-
dation to represent the radial and tangential flexibility of the tread rubber. It was assumed that
the wheel-body was fixed in space and and the wheel was only allowed to rotate at a constant
speed. These authors investigated the contact problem with and without sliding and considered
different damping models, including structural damping, where the damping coefficient is in-
versely proportional to the angular velocity of the tyre. It was found that the rolling resistance of
a driven wheel and a driving wheel were almost identical and that the rolling resistance on a flat
and dry road arose mainly from hysteresis of the tyre; sliding contributed very little to the rolling
resistance. When no sliding occurred, they derived a unique definition for the effective rolling
radius and found that it had a small dependency on the coefficient of friction. Structural damping
was found to be more adequate than viscous damping as rolling resistance was generally found
to vary very little with speed. Comparison with experimental results was only provided for the
effective rolling radius.
1.4 Aim of the project

As previously mentioned, a substantial amount of fuel is being consumed to overcome rolling resistance and significant economic and environmental improvements can be obtained by reducing fuel consumption. It is therefore desirable to have a good understanding of the mechanisms of rolling resistance. Although a large volume of research exist on steady rolling resistance of tyres under constant vertical load, very few studies have considered the energy dissipation in the tyre and suspension together. Most of the existing literature is concerned with car tyres and little data is available on truck tyres, which have different designs and carry substantially higher loads.

Furthermore, no laboratory measurements of truck tyre rolling resistance under dynamic vertical load have been reported until recently: Popov et al. [18] showed that there was no significant effect of dynamic load on mean rolling resistance (due to static vertical load), whether on a smooth drum or in the presence of cleats (to simulate a rough road profile) in the frequency range considered. The measurements were made for two types of truck tyres on a large drum and the dynamic vertical force was generated using a hydraulic actuator. It was found that accurate measurements were very difficult to obtain and it was necessary to correct for both the longitudinal and pitch displacement of the wheel hub, as well as the force components resulting from acceleration of the mass outboard of the load cell. LVDTs and accelerometers were used for this purpose [17, 18].

The aim of this project is to apply and possibly modify existing tyre models to predict rolling resistance under dynamic vertical load in the frequency range of interest (0–20 Hz) and compare the results with the available experimental data from [18]. As it was shown that the presence of cleats does not affect the rolling resistance coefficient, they will not be considered here. However, it should be possible to use or extend the models developed here to predict rolling resistance for cleat tests as well.

1.5 Outline of the report

Chapter 2 derives the equations for the tyre models considered and modifies them if required for rolling resistance calculations under dynamic vertical load. Nomenclature for the various models is detailed on pages v–viii. In addition to the flexible ring model [4], which is used throughout this report, three models are considered [13, 21, 27] with different approaches to varying vertical load and tyre-road interface. The numerical values for the model parameters are evaluated in a systematic and rigorous way and applied to truck tyres.

Chapter 3 compares the model predictions with the available experimental data from [18] in the low frequency range (0–10 Hz) and extends the analysis to a larger frequency range (0–50 Hz) in order to evaluate whether or not the different tyre models are able to capture the essential features of tyre dynamics and tyre-road contact mechanics. The underlying model assumptions and the influence of key model parameters are discussed.

Finally, the main conclusions are drawn in Chapter 4 and guidelines are issued as to which model is best suited for rolling resistance calculations under dynamic vertical load.
Chapter 2

Tyre models considered

2.1 Flexible ring model

The flexible ring tyre model was firstly introduced by Tielking [23], and later on Klingbeil made a significant contribution to it in [8]. The formulation used in this report is that developed by Gong [4].

In this model, the tread band structure of a typical tyre is treated as a thin circular ring (Figure 2.1), restrained at its inner surface, both in the radial and circumferential directions, by a continuous annulus of spring-damper elements. An auxiliary elastic foundation with spring elements is attached to the outer ring surface to represent the radial and tangential flexibility of tread rubber [4, 7].

Figure 2.1: The flexible ring model [4, 7].
2.1.1 Equations of motion [4]

The dynamics of the rotating tread band is more easily analysed in a coordinate system which rotates with the wheel, whereas for the analysis of contact deformations and forces, it is more convenient to use a coordinate system which translate with the contact patch.

A point on the tyre is thus located by its polar coordinates \((r, \phi)\) in the non-rotating coordinate system with Cartesian coordinates \((x, z)\), and \((r, \theta)\) in the rotating coordinate system with Cartesian coordinates \((x^{*}, z^{*})\). The two coordinate systems are related through:

\[
x^{*} = x \cos(\Omega t) + z \sin(\Omega t) \\
z^{*} = -x \sin(\Omega t) + z \cos(\Omega t) \\
\theta = \phi - \Omega t.
\]

The tread band is assumed to behave as an inextensible curved beam which bends according to the Bernoulli-Euler assumption. The radial and tangential displacements \(w\) and \(v\) at any point on the middle surface of the inextensible ring are related by:

\[
w = -\frac{\partial v}{\partial \theta}.
\]

The rotation angle \(\beta\) of the tread band cross-section is:

\[
\beta = \frac{1}{R} \left( v - \frac{\partial w}{\partial \theta} \right).
\]

The equations of motion of the tread band and the wheel, in the rotating coordinate system, are expressed in terms of \(v\):

\[
-I_{r} \ddot{\theta}_{r} + 2\pi k_{w} R^{3} \theta_{r} - R^{2} \int_{0}^{2\pi} k_{w} v \, d\theta = t \quad (2.1b)
\]

\[
m(\ddot{x} - 2\Omega \dot{x} - \Omega^{2} x) + \pi R(k_{w} + k_{v})x = \int_{0}^{2\pi} (-q_{v} \cos \theta - q_{w} \sin \theta) \, d\theta = f_{x} \quad (2.1c)
\]

\[
m(\ddot{z} + 2\Omega \dot{z} - \Omega^{2} z) + \pi R(k_{w} + k_{v})z = \int_{0}^{2\pi} (-q_{v} \sin \theta + q_{w} \cos \theta) \, d\theta = f_{z}. \quad (2.1d)
\]

The dot denotes differentiation with respect to time; \(\sigma_{0\theta}^{0}\) is the initial stress in the tread band due to the action of the centrifugal force and inflation \(p_{0}\) and is given by:

\[
\sigma_{0\theta}^{0} A = p_{0} b R + \rho AR^{2} \Omega^{2};
\]
and \( I_r \) are the mass and moment of inertia of the wheel without tyre; \( x^* \) and \( z^* \) are the wheel displacements along the rotating axes; \( \theta_r \) is the mean angular displacement of the tread band with respect to the wheel hub which may be described as a windup rotation by the application of a torque \( t \); the forces \( f_{x^*} \) and \( f_{z^*} \) act at the wheel axle; the external distributed forces and moments acting on the ring at the contact area are \( q_w, q_v, \) and \( q_\beta \).

To study the vibration transmission properties of the tyre, the solution is expressed as a weighted summation of the modes of the system:

\[
v(\theta, t) = \sum_{n=0}^{\infty} [a_n(t) \cos(n\theta) + b_n(t) \sin(n\theta)]. \tag{2.2}
\]

This has the effect of decoupling the equations of motion (2.1a)–(2.1d) into a set of second order ordinary differential equations in time, and leads to a spatial model with mass, stiffness and damping-gyroscopic matrices \([M_n], [K_n], [G_n]\), and generalised vector of external forces \( f_n \):

\[
M_n \ddot{u}_n + G_n \dot{u}_n + K_n u_n = f_n. \tag{2.3}
\]

There are several sets of matrices and vectors, depending on the mode number \( n \). For \( n = 0 \),

\[
M_0 = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_r \end{bmatrix}, \quad G_0 = \begin{bmatrix} c_0 & g_0 & 0 \\ -g_0 & c_0 & 0 \\ 0 & 0 & c_r \end{bmatrix}, \quad K_0 = \begin{bmatrix} k_0 & 0 & k_{0r} \\ 0 & k_0 & 0 \\ k_{0r} & 0 & k_r \end{bmatrix},
\]

\[
u_0 = \begin{bmatrix} a_0 \\ b_0 \\ \theta_r \end{bmatrix}^T, \quad f_0 = \begin{bmatrix} \xi_0 \\ \eta_0 \\ \frac{t}{2\pi R} \end{bmatrix}^T.
\]

For \( n = 1 \),

\[
M_1 = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_a & 0 \\ 0 & 0 & 0 & m_a \end{bmatrix}, \quad G_1 = \begin{bmatrix} c_1 & g_1 & 0 & 0 \\ -g_1 & c_1 & 0 & 0 \\ 0 & 0 & c_a & -g_a \\ 0 & 0 & g_a & c_a \end{bmatrix}, \quad K_1 = \begin{bmatrix} k_1 & 0 & 0 & -k_{12} \\ 0 & k_1 & k_{23} & 0 \\ 0 & k_{23} & k_a & 0 \\ -k_{12} & 0 & 0 & k_a \end{bmatrix},
\]

\[
u_1 = \begin{bmatrix} a_1 \\ b_1 \\ x^* \\ z^* \end{bmatrix}^T, \quad f_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \\ \frac{f_{x^*}}{\pi R} \\ \frac{f_{z^*}}{\pi R} \end{bmatrix}^T,
\]

while for \( n > 1 \),

\[
M_n = \begin{bmatrix} m_n & 0 & 0 \\ 0 & m_n & 0 \\ 0 & 0 & m_n \end{bmatrix}, \quad G_n = \begin{bmatrix} c_n & g_n \\ -g_n & c_n \end{bmatrix}, \quad K_n = \begin{bmatrix} k_n & 0 \\ 0 & k_n \end{bmatrix},
\]

\[
u_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}^T, \quad f_n = \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix}^T.
\]

The contact forces in the load vector are determined by integration of the external distributed forces and moments:

\[
\xi_n = \frac{1}{\pi} \int_0^{2\pi} \left( q_v + \frac{\partial q_w}{\partial \theta} \right) \cos(n\theta) \, d\theta \tag{2.4a}
\]

\[
\eta_n = \frac{1}{\pi} \int_0^{2\pi} \left( q_v + \frac{\partial q_w}{\partial \theta} \right) \sin(n\theta) \, d\theta \tag{2.4b}
\]
The coefficients of the matrices in both the rotating and non-rotating coordinate systems have the same general appearance and only slight differences in the specific element values. Since the non-rotating coordinate system is more appropriate for the study of the vibration transmission properties of tyres, only those elements are given here:

\[ m_n = \rho A \left( 1 + n^2 \right), \quad m_r = \frac{I_r}{2\pi R}, \quad m_a = \frac{m}{\pi R}, \quad g_n = 2n\rho A\Omega \left( n^2 - 1 \right), \quad g_a = 0, \]

\[ k_n = \frac{E I}{R^3} n^2 + \frac{\sigma_{00} A}{R^2} \left( 1 - n^2 \right)^2 - \frac{p_0 b}{R} \left( 1 - n^2 \right) + k_v + k_w n^2 - \rho A\Omega^2 \left( 1 - n^2 \right)^2, \]

\[ k_r = k_v R^2, \quad k_{0r} = -k_v R, \quad k_a = k_{12} = k_{23} = k_w + k_v. \]

The damping coefficients depend on the type of damping model chosen. For example, viscous damping is introduced through the damping ratios \( \lambda_n, \lambda_r \) and \( \lambda_a \):

\[ c_n = 2\lambda_n \sqrt{m_n k_n}, \quad c_r = 2\lambda_r \sqrt{m_r k_r}, \quad c_a = 2\lambda_a \sqrt{m_a k_a}. \]

Other types of damping can be considered, but this will have little or no effect for the calculations developed here in the frequency range of interest (0–20 Hz).

### 2.1.2 Model parameters

The parameters used for numerical simulations are given in Table 2.1. They are for a Dunlop tyre SP241 385/65R22.5.

<table>
<thead>
<tr>
<th>Model parameter</th>
<th>Notation</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>wheel body mass</td>
<td>( m )</td>
<td>23</td>
<td>kg</td>
</tr>
<tr>
<td>wheel moment of inertia</td>
<td>( I_r )</td>
<td>17.72</td>
<td>kgm(^2)</td>
</tr>
<tr>
<td>ring radius</td>
<td>( R )</td>
<td>0.518</td>
<td>m</td>
</tr>
<tr>
<td>ring width</td>
<td>( b )</td>
<td>0.28</td>
<td>m</td>
</tr>
<tr>
<td>ring mass density</td>
<td>( \rho A )</td>
<td>16.5</td>
<td>kgm(^{-1})</td>
</tr>
<tr>
<td>ring bending stiffness</td>
<td>( EI )</td>
<td>2.80</td>
<td>Nm(^2)</td>
</tr>
<tr>
<td>sidewall radial stiffness</td>
<td>( k_w )</td>
<td>(2.82 \times 10^6)</td>
<td>Nm(^{-2})</td>
</tr>
<tr>
<td>sidewall tangential stiffness</td>
<td>( k_v )</td>
<td>(3.80 \times 10^5)</td>
<td>Nm(^{-2})</td>
</tr>
<tr>
<td>tread rubber thickness</td>
<td>( h_0 )</td>
<td>0.02</td>
<td>m</td>
</tr>
<tr>
<td>tread rubber radial stiffness</td>
<td>( k_{Et} )</td>
<td>7.65 \times 10^7</td>
<td>Nm(^{-2})</td>
</tr>
<tr>
<td>tread rubber tangential stiffness</td>
<td>( k_{Gt} )</td>
<td>3.06 \times 10^7</td>
<td>Nm(^{-2})</td>
</tr>
</tbody>
</table>

Table 2.1: Parameter values for the flexible ring model of Dunlop tyre SP241 385/65R22.5.

The three stiffness parameters \( EI, k_w \) and \( k_v \) were estimated by using optimisation routines in order to minimise the error between the calculated natural frequencies and the measured ones through modal analysis of the tyre [12] (the first six modes of vibration were considered). The measurements showed that the viscous damping ratio was approximately constant over all modes of vibration, giving \( \lambda_n = \lambda_r = \lambda_a = 1.5\% \). The resulting velocity (mobility) transfer function for a driving excitation point using viscous damping with an inflation pressure of 6.5 bar is shown in Figure 2.2.
Although the first mode at approximately 50 Hz is missing in the measured transfer function, the theoretical transfer function matches the experimental one very well, both in terms of magnitude and resonances for the first few modes. The reason why the first mode is missing in the measured transfer function is because it is a translational mode, where the belt moves like a rigid ring, and it can only be measured with the wheel fixed [22]. This was not the case during the experimental modal analysis of the Dunlop tyre, since the tyre was freely suspended by four chains in a horizontal position. For further details, please refer to [12].

Geng [2, 3] repeated the analysis for a similar tyre (295/80R22.5) but considered more complex damping models. In his experimental work, the wheel was fixed so that Geng was able to measure the first (translational) mode of vibration. Using a sophisticated optimisation procedure, he obtained similar values for $k_w$. However, the bending stiffness $EI$ he found was approximately 10 times less than the value used here. Also, he found a strong dependency of $k_v$ on inflation pressure, whereas all other parameters were relatively constant. At identical inflation pressure, his value of $k_v$ is very similar to the value used here. The discrepancy in the bending stiffness has little influence on the simulations for rolling resistance performed in this report.

The tread rubber radial stiffness $k_{Et}$ was estimated by using test data for the treading material provided by Dunlop Tyres Ltd [16, 17]. The tangential stiffness $k_{Gt}$ is more difficult to evaluate. Kim and Savkoor [7] used a value which was approximately 40% of $k_{Et}$. The same ratio was used here. The mass and moment of inertia of the wheel were measured [12]. The average rotating speed of the wheel-tyre system $\Omega$ is given by $\Omega = V_x/R$, where $V_x$ is the forward speed (assumed to be constant).
2.1.3 Vibration transmission properties

A transfer function matrix between the contact patch and the wheel can be derived for the free tyre-wheel system using the equations of motion of the flexible ring model and assuming a point contact between tyre and road [4]:

\[
\begin{bmatrix}
V(s) \\
W(s) \\
\Theta_v(s) \\
X(s) \\
Z(s)
\end{bmatrix} =
\begin{bmatrix}
H_{11} & H_{12} & H_{13} & H_{14} & 0 \\
H_{21} & H_{22} & 0 & 0 & H_{25} \\
H_{31} & 0 & H_{33} & 0 & 0 \\
H_{41} & 0 & 0 & H_{44} & 0 \\
0 & H_{52} & 0 & 0 & H_{55}
\end{bmatrix}
\begin{bmatrix}
P_v(s) \\
P_w(s) \\
T(s) \\
F_x(s) \\
F_z(s)
\end{bmatrix}
\]

where the capital letters represent the Laplace transform of the time-dependent variables: longitudinal and vertical displacements \(v\) and \(w\) at the contact point; dynamic angular displacement \(\theta_r\) of the wheel due to speed variations about the angular velocity \(\Omega\); longitudinal and vertical displacement \(x\) and \(z\) of the wheel hub; and the corresponding forces \(p_v\) and \(p_w\) at the contact point, and torque and forces \(t\), \(f_x\) and \(f_z\) at the wheel centre. The terms in the matrix are given by:

\[
H_{11} = \frac{1}{\pi R} \left[ 0.5 t_{11}^{11} + \sum_{n=1}^{+\infty} t_{11}^{11} \right]
\]

\[
H_{12} = -\frac{1}{\pi R} \sum_{n=1}^{+\infty} [n^2 t_{12}^{12}]
\]

\[
H_{13} = \frac{1}{2\pi R} t_{13}^{13} = H_{51}
\]

\[
H_{14} = \frac{1}{\pi R} t_{14}^{14} = H_{41}
\]

\[
H_{22} = \frac{1}{\pi R} \sum_{n=1}^{+\infty} [n^2 t_{22}^{11}]
\]

\[
H_{25} = -\frac{1}{\pi R} t_{14}^{14} = H_{52}
\]

\[
H_{33} = \frac{1}{2\pi R} t_{33}^{33} = H_{55}
\]

\[
H_{44} = \frac{1}{\pi R} t_{33}^{33} = H_{55}
\]

For numerical calculations, 30 modes have been included in the summations after convergence studies. The elements \(t_{ij}^{nm}\) \((n = 0, 1, \ldots \) and \(i, j = 1, \ldots , 4\)) of the transfer functions between generalised displacements and forces are given by:

\[
t_{11}^{11} = \frac{-m_r \omega^2 + c_r i \omega + k_r}{(-m_0 \omega^2 + c_0 i \omega + k_0)(-m_r \omega^2 + c_r i \omega + k_r) - k_{0r}^2}
\]

\[
t_{13}^{13} = \frac{-k_{0r}}{(-m_0 \omega^2 + c_0 i \omega + k_0)(-m_r \omega^2 + c_r i \omega + k_r) - k_{0r}^2}
\]

\[
t_{33}^{33} = \frac{-m_0 \omega^2 + c_0 i \omega + k_0}{(-m_0 \omega^2 + c_0 i \omega + k_0)(-m_r \omega^2 + c_r i \omega + k_r) - k_{0r}^2}
\]
\[ t_{m1}^{14} = \frac{k_1}{(-m_1\omega^2 + c_1i\omega + k_1)(-m_n\omega^2 + c_ni\omega + k_n) - k_1^2} \]

\[ t_{m1}^{33} = \frac{-m_1\omega^2 + c_1i\omega + k_1}{(-m_1\omega^2 + c_1i\omega + k_1)(-m_n\omega^2 + c_ni\omega + k_n) - k_1^2} \]

\[ t_{mn}^{11} = \frac{-m_n\omega^2 + c_ni\omega + k_n}{(-m_n\omega^2 + c_ni\omega + k_n)^2 + (g_ni\omega)^2} \]

\[ t_{mn}^{12} = \frac{-g_ni\omega}{(-m_n\omega^2 + c_ni\omega + k_n)^2 + (g_ni\omega)^2}. \]

Appropriate boundary conditions must be chosen in order to derive the frequency response functions of the model. Two cases are of particular interest:

1. **Cleat tests**: this corresponds to the case where the tyre is pressed against a rotating drum and small obstacles (cleats) are attached to the drum. Cleat tests are widely used to simulate a vehicle going at a constant speed over rough road. The axle is generally fixed in space so that the boundary conditions in this case are \( X = Z = 0 \).

2. **Axle vibrations**: here, the tyre is preloaded against the drum and the external excitation is applied at the axle, which is free to move in the vertical direction (sometimes axle motion in the horizontal direction is also allowed). Since the tyre is rolling on a smooth surface, no motion of the tread in the contact patch is allowed in the vertical direction. If the friction between the tyre and the drum is sufficiently large, no motion of the tread occurs in the longitudinal direction either. It follows that the boundary conditions in that case are \( V = W = 0 \).

For cleat tests, the transfer function of interest is:

\[ K_{T,1}(s) = \frac{F_z(s)}{W(s)} = \frac{(\alpha_{54}\alpha_{42} - \alpha_{44}\alpha_{52})}{(\alpha_{44}\alpha_{55} - \alpha_{45}\alpha_{54})} \tag{2.6} \]

where the matrix \([\alpha]\) is defined by:

\[
\begin{bmatrix}
P_v \\
P_w \\
\Theta_r \\
X \\
Z \\
\end{bmatrix} = 
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{15} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{51} & \alpha_{52} & \cdots & \alpha_{55} \\
\end{bmatrix} 
\begin{bmatrix}
V \\
W \\
T \\
F_x \\
F_z \\
\end{bmatrix}
\]

It can be obtained by modal synthesis method (or sometimes called receptance method) which consists in partitioning the degrees of freedom (DOFs) between boundary DOFs and remaining DOFs. In that case, it yields:

\[
[\alpha] = \begin{bmatrix}
[H]_{bb}^{-1} & -[H]_{bb}^{-1}[H]_{br} \\
[H]_{rb}^{-1}[H]_{br} & [H]_{rr}^{-1} - [H]_{rb}^{-1}[H]_{bb}^{-1}[H]_{br} \\
\end{bmatrix}
\]

\[
[H]_{bb} = 
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22} \\
\end{bmatrix}
\]
For axle vibrations, the boundary conditions can be applied directly to equation (2.5) to give the transfer function of interest: 

\[ K_{T,2}(s) = \frac{F_z(s)}{Z(s)} = \left[ H_{55} - \frac{H_{52}H_{11}H_{25}}{H_{11}H_{22} - H_{12}H_{21}} \right]^{-1}. \] (2.7)

\( K_{T,1}(s) \) and \( K_{T,2}(s) \) represent the dynamic vertical stiffnesses of the tyre and are used extensively in this report. Figure 2.3 compares these two stiffnesses with a simple model consisting of a spring and viscous dashpot in parallel \((k_t + i\omega c_t)\). The value for the spring stiffness corresponds to the stiffness measured by static deflection of the tyre \((8.3 \times 10^5 \text{ Nm}^{-1})\). A similar value was obtained through a detailed finite element analysis of the contact patch [16]. A damping ratio of 1.5% of critical damping was used for the dashpot, as measured during experimental modal analysis [12]. The static tyre stiffness obtained with the flexible ring model (the same in both cases) is approximately 46% higher than the measured value. This is believed to be due to the contact point assumption, yielding a higher theoretical stiffness. Hysteretic damping was also considered but it proved to have little or no influence over the frequency range of interest, so it is not presented here.

![Figure 2.3: Comparison of dynamic tyre stiffnesses.](image-url)
2.1.4 Power consumption

In order to investigate the influence of tyre and suspension parameters on vehicle rolling resistance, the flexible ring model was incorporated into a quarter-vehicle model in place of the common spring and dashpot (referred to as the ‘enhanced quarter-truck’ model) [12]. The resulting power consumption of both tyre and suspension at a constant speed of 80 km/h for a wide range of suspension damping values and for an ‘average’ road roughness profile is shown in Figure 2.4.

![Figure 2.4: Power consumption at 80 km/h against suspension damping for an average road roughness profile [12].](image)

At low levels of suspension damping, the energy losses are dominated by the losses in the tyre. However, for a typical value of suspension damping ($10^4$ Nsm$^{-1}$), the suspension contributes 76% of the total energy loss (560 W). The energy loss has a maximum for a suspension damping value of approximately 45 Nsm$^{-1}$, at which point the suspension contributes 64% of the total energy loss (2400 W). It should be noted that this is not a realistic value of suspension damping and that for the practical range of suspension damping, the power consumed by both the tyre and the suspension remain approximately constant. For comparison, Segel and Lu [21] report a contribution to rolling resistance of approximately 55% from the suspension at the same speed, but over a ‘very poor’ road.
2.2 Radially distributed hysteretic springs [21]

This method is derived from Segel and Lu’s approach [21]. Their model, shown in Figure 2.5, consists of a quarter-vehicle model, where the viscoelastic properties of the tyre are represented by radially distributed independent springs, each spring behaving as an hysteretic elastic element (i.e. the dynamic spring stiffness as a function of frequency can be written as $K + i\omega a$).

![Tyre model diagram]

Figure 2.5: Segel and Lu’s tyre model for rolling resistance calculations [21].

The energy loss per revolution of the tyre is assumed to be the summation of the energy lost in each hysteretic spring compressed during the rolling. On a smooth and horizontal road, the vertical deflection $\delta_0$ is constant and represents the maximum deflection that each hysteretic spring experiences as it passes through the contact patch. The area of the stress-strain hysteresis loop is proportional to the magnitude squared [11] so that the energy lost in each spring element per revolution of the tyre is given by:

$$\Delta W_t = \frac{1}{2} a \Delta \theta \delta_0^2$$
where:

\[ a \text{ [Nm}^{-1}\text{rad}^{-1}] = \text{constant of proportionality, called the ‘hysteresis coefficient’ per unit angle around the tyre;} \]

\[ \Delta \theta \text{ [rad]} = \text{angular dimension over which the continuous elasticity of the tyre (in the radial direction) is lumped;} \]

\[ \delta_0 \text{ [m]} = \text{static deflection of the tyre and also maximum deflection seen by each spring element;} \]

\[ \Delta W_t \text{ [Nm]} = \text{energy or work consumed by each spring element per tyre revolution.} \]

Therefore, the total energy consumed by the tyre per revolution is:

\[
W_t = \int_0^{2\pi} \frac{1}{2} a \delta_0^2 d\theta = \frac{a}{2R_e} \delta_0^2 2\pi R_e
\]

where \( R_e \text{ [m]} \) is the effective rolling radius of the tyre. Segel and Lu assumed that \( R_e \) was equal to the radius of the undistorted tyre (\( R_0 \)). In reality, \( R_e \) is smaller than \( R_0 \) and varies according to the rolling conditions, the ground, the obstacles, etc. When travelling over a perfectly smooth road, the following approximation can be used [24]:

\[
R_e \simeq 0.98 R_0.
\] (2.8)

Given that the energy consumed by the tyre in one revolution is equal to the work required to move the tyre forward by a distance corresponding to one revolution, the following relationship holds:

\[
2\pi R_e F_x = W_t = \frac{a}{2R_e} \delta_0^2 2\pi R_e
\]

so that:

\[
F_x = \frac{a}{2R_e} \delta_0^2
\] (2.9)

where \( F_x \text{ [N]} \) is the rolling resistance derived from the energy consumed during the rolling cycle.

Equation (2.9) is as per Segel and Lu’s original findings [21] and is normally valid only for steady state calculations over a smooth road. It is proposed here to use this formulation to predict rolling resistance under dynamic vertical load over a smooth road. In order to do that, the static deflection of the tyre \( \delta_0 \) is calculated using the dynamic tyre stiffness shown in Figure 2.3.

By doing so, it effectively becomes a dynamic deflection:

\[ \delta_0(\omega) = \frac{F_z(\omega)}{K_{Tz}(\omega)} \]

so that the dynamic rolling resistance is given by:

\[
\frac{F_x(\omega)}{F_z(\omega)} = \frac{a}{2R_e} \frac{F_z(\omega)}{K_{Tz}(\omega)}.
\] (2.10)

The difficulty here is to find a suitable value for the hysteresis coefficient \( a \). Both the sidewalls and the tread band are involved in the energy losses in the tyre, so that both have to be considered. As \( k_w \) and \( k_{Et} \) in Figure 2.1 are in series, the following expression is proposed for \( a \):

\[ a = \left( \frac{1}{k_w} + \frac{1}{k_{Et}} \right)^{-1} \frac{R_0}{2\pi} \]
giving a numerical value of $2.32 \times 10^5 \text{ Nm}^{-1}\text{rad}^{-1}$. The same approach can be used for a standard spring and dashpot in parallel, in which case:

$$
\frac{F_x(\omega)}{F_z(\omega)} = \frac{a}{2Re} \frac{F_z(\omega)}{(k_t + i\omega c_t)^2}
$$

and

$$
a = \frac{k_t}{2\pi}. \quad (2.11)
$$

The results are presented in Chapter 3, together with the laboratory measurements.

### 2.3 Pacejka’s transient tyre model [13]

In his very extensive book on tyre dynamics [13], Pacejka devotes one chapter to the application of single contact point transient tyre models, in which he considers the response of the axle forces $F_x$ and $F_z$ to in-plane axle motions ($x, z$), road waviness and tyre non-uniformities.

The relationship between normal load $F_z$ and radial tyre deflection $\rho$ is simplified using the radial stiffness $C_{Fz}$ ($F_z = C_{Fz}\rho$). For a non-steady state analysis, the differential equation between the fore and aft deflection of the tyre $u$ holds:

$$
\frac{du}{dt} + \frac{1}{\sigma_\kappa}Vu = V\kappa = -V_{sx}
$$

where $V$ is the forward speed, $\kappa$ is the longitudinal wheel slip ratio, $V_{sx}$ is the longitudinal slip velocity and $\sigma_\kappa$ is the relaxation length. The last quantities are defined as follows:

$$
\begin{align*}
V_{sx} &= V - r_e\Omega \\
\kappa &= \frac{V_{sx}}{V} \\
\sigma_\kappa &= \frac{C_{F\kappa}}{C_{Fx}}
\end{align*}
$$

where $r_e$ is the effective rolling radius, $\Omega$ is the wheel speed of revolution, $C_{F\kappa}$ is the longitudinal slip stiffness and $C_{Fx}$ the longitudinal tyre stiffness. The longitudinal force $F_x$ is obtained by multiplying the deflection $u$ by the longitudinal stiffness $C_{Fx}$. This is illustrated in Figure 2.6.

In general, the effective rolling radius $r_e$ changes with tyre deflection $\rho$:

$$
r_e = r_f - f(\rho)
$$

where $r_f$ is the free (undeformed) radius, which may vary along the tyre circumference due to tyre non-uniformities. The loaded radius $r$ can be expressed as:

$$
r = r_f - \rho$$
When looking at a graph of \( f(\rho) \), two quantities can be defined: the slope \( \eta \), which indicates the influence of changes in tyre deflection, and \( \varepsilon \), which indicates the influence of tread depth near the nominal load. The slope \( \eta \) is estimated to be equal to approximately 0.1 for radial car tyres and 0.4 for bias-ply car tyres [13]. In order to perform a linear analysis, small deviations from the undisturbed condition are assumed (indicated with a tilde):

\[
\begin{align*}
\hat{r}_e &= r_{e,0} + \hat{r}_e \\
r &= r_0 + \hat{r} \\
\rho &= \rho_0 + \hat{\rho}.
\end{align*}
\]

Variations of the free radius \( r_f \) may occur along the circumference of the tyre due to variations of the carcass radius \( r_c \) and other variations of the tread thickness \( d_t \); \( \hat{r}_f = \hat{r}_c + \hat{d}_t \).

The following linear relationship can be derived for the variations in effective rolling radius:

\[
\hat{r}_e = \hat{r}_f - \varepsilon \hat{d}_t - \eta \hat{\rho} = \hat{r}_c + (1 - \varepsilon) \hat{d}_t - \eta \hat{\rho}.
\]

The horizontal force response has three components, as illustrated in Figure 2.7:

- the horizontal component of the normal load,
- the variation of the rolling resistance,
- the longitudinal force response to the variation of the wheel longitudinal slip.
Figure 2.7: In-plane excitation by road roughness and by vertical and fore and aft axle motions. Reproduced from [13].

Therefore, the horizontal in-plane force \( X \), once linearised, can be written as:

\[
X = X_0 + \dot{X} = -F_{r0} + \dot{F}_x + F_{z0} \frac{dw}{ds} = -F_{r0} + \ddot{F}_r + \ddot{F}_\kappa + F_{z0} \frac{dw}{ds},
\]

where \( F_{r0} \) is the average rolling resistance force. Pacejka assumed that the variation in the rolling resistance force was directly transmitted to the tread through changes in vertical load

\[
\ddot{F}_r = A_r \ddot{F}_z,
\]

where the variation in vertical load results from variations in deflection and possibly changes in the radial stiffness along the circumference of the tyre:

\[
\ddot{F}_z = C_{Fz0} \ddot{\rho} + \rho_0 \dddot{C}_{Fz}.
\]

In terms of variation in the radial static deflection, this can be expressed as \( \ddot{\rho}_s = \rho_0 \frac{\dddot{C}_{Fz}}{C_{Fz0}} \) or \( \ddot{F}_z = C_{Fz0} (\ddot{\rho} - \ddot{\rho}_s) \). The variation in the tangential slip force becomes (with \( \kappa' \) being the transient longitudinal slip):

\[
\ddot{F}_\kappa = C_{F\kappa} \ddot{\kappa}' = C_{F\kappa} \frac{u}{\sigma_\kappa}.
\]

The variations in the forward velocity, speed of revolution and effective rolling radius can be linearised as:

\[
V_{sx} = \ddot{V} - \dot{\Omega}_0 \ddot{\rho}_e - r_0 \ddot{\Omega}.
\]
The variations in tyre deflection result from wheel vertical displacement, tyre out-of-roundness and road height changes:

\[ \tilde{\rho} = z + \tilde{r}_c + \tilde{d}_t - w. \]

The wheel angular velocity is given by the equation of motion:

\[ I_w \dot{\Omega} = -r F_x - M_y \quad (2.12) \]

where \( I_w \) is the wheel polar moment of inertia. \( M_y \) acts about the transverse axis through the contact patch centre \( C \) and is due to rolling resistance and eccentricity of the tread band, which constitutes the first harmonic of the tyre out-of-roundness. This harmonic has a 90 degrees phase lead with respect to the loaded radius variations sensed at the contact centre [13], so that (the subscript 1 denoting the first harmonic):

\[ M_y = M_r + F_z \tilde{r}_1. \]

The rolling resistance moment \( M_r \) is assumed to be directly related to the rolling resistance force \( F_r \), so that the rolling resistance components cancel out in the right-hand side of equation (2.12), which reduces to:

\[ I_w \dot{\Omega} = -r_0 F_r - F_{z0} \tilde{r}_1. \]

Combining all the previous relevant equations and eliminating all the variables, except the input quantities \( \tilde{V} (= \tilde{x}) \), \( z \), \( w \), \( \tilde{r}_c \), \( \tilde{d}_t \), \( \tilde{\rho}_s \), \( \tilde{r}_1 \) (= first harmonic of \( \tilde{r}_f = \tilde{r}_c + \tilde{d}_t \)) and the output \( \tilde{X} \), as well as \( \tilde{r}_c \) and \( \tilde{\rho} \), after taking the Laplace transform, yields [13]:

\[ \tilde{X}(s) = C_{F_F} \frac{I_w \tilde{\Omega}_0 s \tilde{r}_c - I_w s^2 \tilde{x} - F_{z0} r_0 e^{\frac{s}{\tilde{\Omega}_0}} \tilde{r}_1}{\sigma \kappa I_w s^2 + I_w V_0 s + C_{F_F} r_0^2} - A_v C_{F_F} (\tilde{\rho} - \tilde{\rho}_s) + F_{z0} s \frac{w}{V_0}. \]

\[ \tilde{r}_c = \eta (w - z) + (1 - \eta - \varepsilon) \tilde{d}_t + (1 - \eta) \tilde{r}_c \]

\[ \tilde{\rho} = z + \tilde{r}_c + \tilde{d}_t - w. \]

From this general expression, the individual transfer functions can be obtained. In the case of the frequency response to axle motions on smooth and level ground, it results in (with \( X = F_x \) and \( s = i\omega \)):

\[ \frac{\tilde{F}_x}{z} = -A_v C_{F_F} - \frac{\eta C_{F_F}}{r_0} \frac{2 \zeta i \nu}{1 - \nu^2 + 2 \zeta i \nu} \quad (2.13a) \]

\[ \nu = \frac{\omega}{\omega_{\Omega_0}} \quad \text{(non-dimensional frequency)} \quad (2.13b) \]

\[ \zeta = \frac{1}{2} \frac{I_w V}{C_{F_F} r_0^2} = \frac{1}{2} \frac{V}{\sigma \kappa \omega_{\Omega_0}} \quad \text{(speed dependent damping ratio)} \quad (2.13c) \]

\[ \omega_{\Omega_0} = \sqrt{\frac{C_{F_F} r_0^2}{I_w}} = \sqrt{\frac{C_{F_F} r_0^2}{\sigma \kappa I_w}} \quad \text{(natural frequency of the tyre-wheel rotation)} \quad (2.13d) \]
The difficulty here is to select appropriate numerical values for the various parameters: $A_r$, $C_{Fz}$, $C_{Fκ}$, $η$, $r_0$, $I_w$. Out of these, $I_w$ can be obtained directly from the flexible ring model parameters ($I_r$ in Table 2.1) and $C_{Fz}$ is the dynamic tyre stiffness $K_{T,2}(ω)$ defined in equation (2.7). $A_r = 6.41 \times 10^{-3}$ is the steady state rolling resistance coefficient as measured by Popov et al. [18].

The longitudinal slip stiffness $C_{Fκ}$ was measured as a function of the vertical load by Christopher B. Winkler, from the University of Michigan Transportation Research Institute (UMTRI), for a similar tyre [25]. The data is shown in Figure 2.8 where the longitudinal tyre force is plotted as a function of the slip ratio for different vertical loads. The longitudinal slip stiffness is obtained by taking the slopes of each curve in Figure 2.8(a) in the linear region (i.e. slip ratio less than 0.1 in absolute value).

Thorvald [22] measured the longitudinal tyre stiffness as a function of vertical load and inflation pressure for a similar heavy vehicle tyre. Based on his results, the following value was used for $C_{Fz} = 7.66 \times 10^5$ Nm$^{-1}$. $η$ was chosen to be 0.1, as per Pacejka’s recommendations.

The loaded tyre radius can be expressed as a function of the free undeformed tyre radius and the dynamic tyre deflection:

$$r = r_f - \frac{F_z(ω)}{K_{T,2}(ω)} \quad \text{(flexible ring model)} \tag{2.14a}$$
$$r = r_f - \frac{F_z(ω)}{k_l + iωc_l} \quad \text{(spring and dashpot)} \tag{2.14b}$$

The resulting loaded tyre radii are compared with experimental data (obtained by subtracting the measured tyre deflection to the free undeformed tyre radius) in Figure 2.9. Both the flexible ring model and the spring-dashpot arrangement give a reasonable estimate at low frequencies, the flexible ring model matching the experimental data better. At high frequencies, only the flexible ring model is able to show the effects of the axle hop resonance at approximately 25 Hz.
Figure 2.9: Loaded tyre radius as a function of the excitation frequency.

Finally, in order to obtain the rolling resistance coefficient from equation (2.13), the
dynamic tyre stiffness $C_{Fz} (= k_t + i\omega c_t$ for the simple spring and dashpot arrangement and
$\tilde{K}_{T,2}(\omega)$ for the flexible ring) is used:

$$\frac{\tilde{F}_x}{F_z} = \frac{\tilde{F}_r}{z} \frac{z}{\tilde{F}_z} = \frac{\tilde{F}_x}{z} \frac{1}{C_{Fz}}$$

(2.15)

The results are presented in Chapter 3, together with the laboratory measurements.

### 2.4 Brush model and contact patch length [27]

Zegelaar derived analytically the response to small variations of vertical load using a brush model [27]. He assumed small values of longitudinal slip so that all the brush elements in the contact zone adhered to the road surface. He modelled the effect of dynamic vertical load as variations in the length of the contact patch so that half the contact patch length $a$ is written as a small variation $\tilde{a}(x)$ on top of a stationary value $a_0$:

$$a = a_0 + \tilde{a}(x)$$

where $x(t)$ is the travelled distance. This is a valid assumption, since the forward velocity $V_{\text{car0}}$ is assumed to be constant. The deformation $u$ of a tread element is then proportional to the time it has spent in the contact patch.
The following quantities are defined and written as small variations (denoted by a tilde) on top of a constant value (denoted by an additional subscript 0):

- $V_{cx} = V_{cx0}$: forward velocity in the contact patch, assumed to be constant,
- $V_{cr} = V_{cr0} + \bar{V}_{cr}$: rolling velocity in the contact patch,
- $V_{c,sx} = V_{cx} - V_{cr} = V_{c,sx0} + \bar{V}_{c,sx}$: longitudinal slip velocity,
- $\kappa_c = -\frac{V_{c,sx}}{V_{cx}}$: practical longitudinal slip,
- $\zeta_{cx} = -\frac{V_{c,sx}}{V_{cr}}$: theoretical longitudinal slip.

The influence of varying contact length on the longitudinal force is more easily analysed by considering the diagram of travelled distance $x$ versus brush position $s$ shown in Figure 2.10. The brush element at time $t$ and position $s$ started at the front edge of the contact patch at time $t - \tau$. The positions of this element during the interval $\tau$ are indicated by the bold line. The longitudinal deformation $u$ can be expressed as a function of the travelled distance $x(t)$ or as a function of the position $s$ in the contact patch [27]:

\[
\begin{align*}
  u(x, \tau) &= [x(t) - x(t - \tau)] \frac{\zeta_{cx}}{1 - \zeta_{cx}} \\
  u(x, \tau) &= \{a [x(t - \tau)] - s\} \zeta_{cx}
\end{align*}
\]

The time interval $\tau$ can be linearised as $\tau = \tau_0 + \tilde{\tau}$ where $\tau_0$ is due to constant slip and constant contact length $a_0$ and is given by:

\[
\tau_0 = a_0 - \frac{s}{V_{cr0}}
\]

The small variations $\tilde{\tau}$ cannot be neglected, but can also be expressed through $\xi$ which is the travelled distance during the time interval $\tilde{\tau}$: $\xi = V_{x} \tilde{\tau}$. The displacement $x$ at time $t - \tau_0$ can be rewritten as a function of the displacement at time $t$ and the position in the contact patch $s$:

\[
x(t - \tau_0) = x(t) - \tau_0 V_{cx} = x(t) - \frac{a_0 - s}{V_{cr0}} V_{cx} = x(t) - (a_0 - s) (1 - \zeta_{cx})
\]

The tread deformations can now be rewritten as:

\[
\begin{align*}
  u(x, \tau) &= [V_{cx} \tau_0 + \xi] \frac{\zeta_{cx}}{1 - \zeta_{cx}} \\
  u(x, \tau) &= \{a [x(t - \tau_0) - \xi] - s\} \zeta_{cx}
\end{align*}
\]
Assuming that $\tilde{a}(x)$ remains small, the distance $\xi$ is also small and the previous equation can be expanded in a Taylor series. Substituting the expression for $x(t - \tau_0)$ and linearising the length of the contact patch in $\xi$ yields [27]:

$$a(x(t - \tau_0) - \xi) \simeq a_0 + \tilde{a} \left[ x(t) - (a_0 - s) \left( 1 - \zeta_{cx} \right) \right] - \xi \tilde{a}' \left[ x(t) - (a_0 - s) \left( 1 - \zeta_{cx} \right) \right]$$

where $\tilde{a}'(x)$ is the derivative of the contact length variation $\tilde{a}(x)$ with respect to the travelled distance $x$. $\xi$ can now be solved as:

$$\left( V_{cx} \tau_0 + \xi \right) \frac{1}{1 - \zeta_{cx}} = a_0 + \tilde{a} \left[ x(t) - (a_0 - s) \left( 1 - \zeta_{cx} \right) \right] - \xi \tilde{a}' \left[ x(t) - (a_0 - s) \left( 1 - \zeta_{cx} \right) \right] - s$$

which gives after linearisation:

$$\xi \simeq \left( 1 - \zeta_{cx} \right) \tilde{a} \left[ x(t) - (a_0 - s) \left( 1 - \zeta_{cx} \right) \right].$$

The longitudinal force is obtained by integrating the tread deformation $u$ over the contact length after multiplying by the tread stiffness per unit length $k_{cp}$:

$$F_{ cx } = \int_{s=-a-\tilde{a}(x)}^{s=a+\tilde{a}(x)} k_{cp} u(t, s) \, ds$$

$$= k_{cp} \zeta_{cx} \int_{s=-a-\tilde{a}(x)}^{s=a+\tilde{a}(x)} \left\{ a_0 - s + \tilde{a} \left[ x(t) + (s - a_0) \left( 1 - \zeta_{cx} \right) \right] \right\} \, ds$$

$$= k_{cp} \zeta_{cx} \left\{ a_0 s - s^2 + \frac{1}{1 - \zeta_{cx}} \tilde{A} \left[ x(t) + (s - a_0) \left( 1 - \zeta_{cx} \right) \right] \right\} \mid_{s=-a_0-\tilde{a}(x)}^{s=a_0+\tilde{a}(x)}$$
where $\tilde{A}(x)$ is the integral of $\tilde{a}(x)$ with respect to the travelled distance $x$:

$$\tilde{A}(x) = \int \tilde{a}(x) \, dx$$

Some further simplifications are introduced:

$$\tilde{A}(x + \tilde{a}(x)) \simeq \tilde{A}(x) + \tilde{a}(x) \tilde{a}(x) \simeq \tilde{A}(x)$$

$$\tilde{A}(x - 2a_0 - \tilde{a}(x)) \simeq \tilde{A}(x - 2a_0) - \tilde{a}(x) \tilde{a}(x) \simeq \tilde{A}(x - 2a_0)$$

This gives the following expression for the longitudinal force:

$$F_{cx} = F_{cx0} + \tilde{F}_{cx} \quad (2.16a)$$

$$F_{cx0} = 2k_{cp}a_0^2\zeta_{cx} \quad (2.16b)$$

$$\tilde{F}_{cx} = 2a_0k_{cp}\tilde{a}(x)\zeta_{cx} + \frac{1}{1 - \zeta_{cx}}k_{cp}\zeta_{cx} \{ \tilde{A}(x) - \tilde{A}[x - 2a_0 (1 - \zeta_{cx})] \} \quad (2.16c)$$

The transfer function is obtained by taking the Laplace transform of the above equation:

$$\frac{\tilde{F}_{x}}{\tilde{a}}(\omega) = 2a_0k_{cp}\zeta_{cx} \left( 1 + \frac{1 - e^{-2i\omega a_0 V_{cr0}}}{2i\omega a_0 V_{cr0}} \right) = \frac{F_{x0}}{a_0} \left( 1 + \frac{1 - e^{-2i\omega a_0 V_{cr0}}}{2i\omega a_0 V_{cr0}} \right) \quad (2.17)$$

The difficulty here is to express the contact length $a$ as a function of the vertical load $F_z$. Zegelaar suggests using a second order polynomial in the square root of the vertical load [27], derived by assuming an elliptic contact patch:

$$a = q_{a2}\sqrt{F_z^2} + q_{a1}\sqrt{F_z}$$

Linearising the vertical load as $F_z = F_{z0} + \tilde{F}_z$ gives:

$$a_0 = q_{a2}\sqrt{F_{z0}^2} + q_{a1}\sqrt{F_{z0}} \quad (2.18a)$$

$$a = q_{a2} \left( F_{z0} + \tilde{F}_z \right) + q_{a1}\sqrt{F_{z0}} \left[ 1 + \frac{\tilde{F}_z}{F_{z0}} \right] \quad (2.18b)$$

so that:

$$\frac{\tilde{a}}{F_z} = q_{a2} + \frac{q_{a1}}{2\sqrt{F_{z0}}} \quad (2.19)$$

Combining equations (2.17) and (2.19) gives the desired expression for the rolling resistance as a function of frequency:

$$\frac{\tilde{F}_x}{F_z}(\omega) = \frac{\tilde{F}_x}{\tilde{a}} \frac{\tilde{a}}{F_z} = 2a_0k_{cp}\zeta_{cx} \left( q_{a2} + \frac{q_{a1}}{2\sqrt{F_{z0}}} \right) \left( 1 + \frac{1 - e^{-2i\omega a_0 V_{cr0}}}{2i\omega a_0 V_{cr0}} \right). \quad (2.20)$$
The parameters required are: $a_0$, $k_{cp}$, $\zeta_{cx}$, $q_{a1}$, $q_{a2}$, $F_{z0}$ and $V_{cr0}$. Both $F_{z0}$ and the constant forward velocity $V_{cx0}$ are known from the experimental conditions. The angular velocity $\Omega$ can therefore be obtained from $V_{cx0}$ and the rolling velocity $V_{cr0}$ is given by the effective rolling radius $r_e$ multiplied by the angular velocity $\Omega$. $\Omega$ should have been available from the experimental data in [18], but difficulties were encountered during the measurements. Knowing $V_{cx0}$ and $V_{cr0}$, the longitudinal slip $\zeta_{cx}$ can then be calculated. The values for $q_{a1}$ and $q_{a2}$ were obtained using the following experimental steady-state data that was measured on a Dunlop tyre SP241 385/65R22.5:

\[
\begin{align*}
F_{z0} &= \begin{bmatrix} 11.26 & 17.96 & 24.92 & 31.90 & 38.51 \end{bmatrix} \text{kN} \\
\delta_0 &= \begin{bmatrix} 4.07 & 11.59 & 19.12 & 26.67 & 34.21 \end{bmatrix} \text{mm}
\end{align*}
\]

where $\delta_0$ is the static tyre deflection. The contact patch length can then be calculated using $\delta_0$, as shown in Figure 2.11:

\[
a_0 = \sqrt{R^2 - (R - \delta_0)^2}
\]

$q_{a1}$ and $q_{a2}$ were obtained by using a least mean square optimisation routine to minimise the error between the theoretical contact length predicted by equation (2.18) and the experimental values. This yielded the following values:

\[
\begin{align*}
q_{a1} &= 3.69 \times 10^{-4} \text{mN}^{-\frac{1}{2}} \\
q_{a2} &= 3.13 \times 10^{-6} \text{mN}^{-1}
\end{align*}
\]

The resulting plot of contact patch length versus vertical load is presented in Figure 2.12. There is a reasonably good agreement between the experimental data points and the theoretical predictions. A value of $2 \times 10^5 \text{N.m}^{-2}$ was chosen for $k_{cp}$. This corresponds approximately to the value given by equation (2.16b) when using the previously derived values for $a_0$ and $\zeta_{cx}$, together with the measured longitudinal force $F_{x0}$ at the appropriate vertical load.
2.5 Flexible ring model with tyre-road interface [7, 26, 27]

Both Zegelaar [26, 27] and Kim and Savkoor [7] considered the flexible ring model with a tyre-road interface at various degrees of complexity. The method adopted by Kim and Savkoor is fairly complex and uses boundary conditions at the front edge of the contact patch which were found to yield numerical complications by previous researchers. The approach presented here is adapted from Zegelaar’s derivation in [26] and [27].

The radial and tangential displacements $v$ and $w$ mentioned in section 2.1 are now expressed for a finite number of contact points $n_e$ and a finite number of vibration modes $n_m$, so that equation (2.2) becomes:

$$v_i = \sum_{n=0}^{n_m} a_n \cos(n\theta_i) + b_n \sin(n\theta_i)$$

$$w_i = -\frac{\partial v_i}{\partial \theta_i} = \sum_{n=0}^{n_m} n a_n \sin(n\theta_i) - n b_n \cos(n\theta_i)$$

where the subscript $i$ denotes the discrete point at an angular position $\theta_i$. In [27], Zegelaar uses discrete points equally space along the ring circumference $\left(\theta_i = \frac{i}{n_e} - 2\pi, \quad i = 1, 2, \ldots, n_e\right)$ whereas in [26], he only considers discrete points in the contact patch:

$$\theta_i = \frac{\pi}{2} + \left(\frac{2i}{n_e} - 1\right) \frac{a}{R}$$

where $a$ is half the contact length (see Figure 2.13). It is worth noting that Zegelaar is using implicitly the small angle approximation.
The vertical and horizontal displacements of the flexible ring model (in the non-rotating coordinate system) are given by:

\[
x_i = -v_i \sin \theta_i + w_i \cos \theta_i \\
z_i = -v_i \cos \theta_i - w_i \sin \theta_i
\]

From these, the vertical and horizontal forces per unit length can be calculated:

\[
q_{x,i} = -k_{Gl} x_i \\
q_{z,i} = -k_{Et} z_i
\]

The terms ‘vertical’ and ‘horizontal’ are used here instead of radial and tangential because the tread element stiffnesses only play a role in the contact patch. The radial and tangential forces acting on the ring are then given by:

\[
q_{v,i} = -q_{x,i} \sin \theta_i - q_{z,i} \cos \theta_i \\
q_{w,i} = q_{x,i} \cos \theta_i - q_{z,i} \sin \theta_i
\]

Combining the above equations gives the radial and tangential forces as a function of the radial and tangential displacements:

\[
q_{v,i} = - \left( k_{Gl} \sin^2 \theta_i + k_{Et} \cos^2 \theta_i \right) v_i + \sin \theta_i \cos \theta_i \left( k_{Gl} - k_{Et} \right) w_i \\
q_{w,i} = \sin \theta_i \cos \theta_i \left( k_{Gl} - k_{Et} \right) v_i - \left( k_{Gl} \sin^2 \theta_i + k_{Et} \cos^2 \theta_i \right) w_i
\]

Figure 2.13: Discrete contact points along the contact patch.
The generalised modal forces in equation (2.4) are now given after discretisation by:

\[
\xi_n = \frac{\Delta \theta}{\pi} \sum_{i=1}^{n_e} \left( q_{v,i} + \frac{\partial q_{w,i}}{\partial \theta_i} \right) \cos(n \theta_i) \\
\eta_n = \frac{\Delta \theta}{\pi} \sum_{i=1}^{n_e} \left( q_{v,i} + \frac{\partial q_{w,i}}{\partial \theta_i} \right) \sin(n \theta_i).
\]

And the modal deformations (after taking the Laplace transform) are obtained by (cf. equation (2.3)):

\[
\mathbf{u}_n = \begin{cases} a_n \\ b_n \end{cases} = \left( M_n s^2 + G_n s + K_n \right)^{-1} \mathbf{f}_n = \left( M_n s^2 + G_n s + K_n \right)^{-1} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix}
\]

To solve the tyre-road contact problem, Zegelaar uses an iteration procedure [27], shown in Figure 2.14. The iteration is stopped when the internal and external pressure distribution are deemed to be sufficiently close. The external pressure distribution is defined as the distributed forces per unit length along the circumference of the tyre, whereas the internal pressure distribution refers to the distributed forces per unit length acting in the tyre sidewalls.

Figure 2.14: Iteration procedure for flexible ring model with tyre-road interface. Reproduced and adapted from [27].
Here, the known inputs are the vertical force $f_z$ applied at the axle and the vertical motion of the axle $z$ (both as a function of the excitation frequency). The longitudinal force $f_x$ and torque $t$ at the axle, together with the corresponding displacement and rotation $x$ and $\theta_r$, can be assumed to be zero. Similarly, the external distributed moment in the contact patch $q_w$ is neglected. The unknown quantities are the distributed forces $q_v$ and $q_w$. Assuming that $q_v$ and $q_w$ can be determined, the longitudinal force in the contact patch are derived by integrating $q_x$ along the length of the contact patch:

$$F_x = \int_{-a}^{+a} q_x(x) \, dx = \sum_{i=0}^{i=n_e} q_{x,i} x_{ci}$$

where $-a \leq x_{ci} \leq +a$ corresponds to the position of the running point along the contact patch.

However, it is not clear how to implement Zegelaar’s iteration procedure, which involves both a sum on the number of contact points ($i$) and the number of vibration modes ($n$) (see Figure 2.14). Furthermore, although the external pressure distribution is clearly defined ($q_v$ and $q_w$), it is unclear what Zegelaar refers to for the internal pressure distribution. For these reasons, this methodology was not pursued any further.

Zegelaar et al. [27,28] also developed a rigid ring tyre model which is thought to be more suited for low frequency analysis, where the zeroth and first order vibration modes dominate. The tyre is modelled as a rigid ring supported on a continuous annulus of sidewall springs, both in the tangential and radial directions, and additional residual stiffnesses represent the deformations in the contact patch. The equations of motions were derived for small variations of the variables on top of the stationary values. A brush model was used to represent the tyre-road interface and derived a transfer function between the longitudinal force and the longitudinal slip velocity. This approach was not adopted here because it was not obvious how to relate the longitudinal slip velocity to the dynamic vertical load for rolling resistance calculations. Also, a number of parameters were required and little insight was given on to how evaluate them.
Chapter 3

Theoretical and experimental results

3.1 Radially distributed hysteretic springs [21]

The model developed by Segel and Lu [21] and adapted in this report to predict rolling resistance under dynamic vertical load (see equations (2.10) and (2.11)) is compared to the available experimental data from [18].

The experimental data was obtained by applying a harmonic dynamic vertical force, in addition to an existing static vertical load $F_{z0} = 35 \text{kN}$ at an inflation pressure of $p_0 = 9.4 \text{ bar}$ and a constant forward speed of $40 \text{ km} \cdot \text{h}^{-1}$. The amplitude of the harmonic component was 50% of the static force. The dynamic vertical load was applied at the following frequencies: 0.5 Hz, 1 Hz, 2 Hz and 4 Hz (higher frequencies were not possible due to the capabilities of the hydraulic actuator). The data was corrected for the curvature of the drum by a factor of $\left(1 + \frac{R_l}{R_D}\right)$ [19] where $R_D$ is the drum radius and $R_l$ is the loaded tyre radius (obtained with the measured tyre deflection).

The results are presented in Figure 3.1, using both the dynamic tyre stiffness of the flexible ring and a standard spring and dashpot arrangement, at low frequencies for comparison with the experimental data and over a wider frequency range. Only viscous damping is considered for both dynamic tyre stiffnesses. The values used for the model parameters in the simulations are detailed in Appendix A.2. The results show that the static level of rolling resistance is better predicted by using a simple spring and dashpot arrangement. This is explained by the discrepancy in static tyre stiffness observed when using the flexible ring model (see Figure 2.3). However, the flexible ring model exhibits a strong resonance at approximately 25 Hz, which is believed to correspond to the axle hop vibration due to tyre resonance. The common spring and dashpot model is unable to capture this important feature of tyre dynamics. Past the resonance, the rolling resistance tends towards zero, which is unrealistic since the mean rolling resistance should not go below the value for zero frequency corresponding to steady rolling. It is difficult to assess what is the influence of dynamic load on rolling resistance because the experimental data is only available for a low frequency range where both models (flexible ring or spring and dashpot) show no or little effect.
Figure 3.1: Dynamic rolling resistance using Segel and Lu’s adapted model.
An important point to note is that the experimental data gives the mean rolling resistance coefficient of the tyre subjected to a static vertical load and a superimposed dynamic vertical load. The model predicts only the dynamic component. Therefore, the comparison between the experimental results and the model predictions must be done with great care. This might explain why the rolling resistance using the dynamic tyre stiffness of the flexible ring model tends towards zero at high frequencies. Finally, the approach of replacing the static tyre deflection by a dynamic one might not be valid as the force-deflection behaviour of the hysteretic spring as it passes through the contact patch is far more complicated than what is presented here, leading to a more complex expression for the energy dissipation.

3.2 Pacejka’s transient tyre model [13]

The model developed by Pacejka [13] and adapted here for truck tyres is compared with the experimental data available in [18]. The same data as previously is used and again both the dynamic tyre stiffnesses derived from the flexible ring model and a simple spring and dashpot arrangement are considered. The values used for the model parameters in the simulations are detailed in Appendix A.3. The results are presented in Figure 3.2.

The static rolling resistance is equally well predicted here by both the flexible ring model and the simple spring and dashpot arrangement. This is because the parameter $A_r$ in equation (2.13), chosen to match the static rolling resistance, was used in both cases. When using a spring and dashpot arrangement for the dynamic tyre stiffness, the results show a resonance at approximately 17 Hz, which corresponds to the natural frequency of the tyre-wheel rotation with respect to the contact patch $\omega_0$. When using the dynamic tyre stiffness from the flexible ring model, a second resonance at 25 Hz corresponding to the axle hop vibration is also shown. In both cases, the mean rolling resistance returns to the static value once past the resonance(s) and the levels of rolling resistance predicted appear to be more realistic than with Segel and Lu’s adapted model.

The results are highly dependent on the value of $\eta$ chosen, which also has a great degree of uncertainty associated. Pacejka does not specify if the estimated value of 0.1 is for truck or car (radial) tyres or both. It is possible that truck tyres, due to their different design and substantially higher loads, will exhibit different values of $\eta$. The effect of $\eta$ on rolling resistance is considered in Fig. 3.3, using the dynamic tyre stiffness derived from the flexible ring model. The results show that increasing $\eta$ has the effect of increasing the amplitude of both resonances. Even when $\eta$ approaches the value recommended by Pacejka, the maximum rolling resistance is still nearly ten times less than the one predicted by Segel and Lu’s model.

When combined with the dynamic tyre stiffness from the flexible ring model, and with suitable numerical values for the different parameters, Pacejka’s approach appears to give very good results as it models both the vertical dynamics of the tyre and the mechanics in the contact patch. It does not suffer the same problem as the previous model, in that it has both a static and a dynamic component. Furthermore, the mean rolling resistance returns to the static value corresponding to steady rolling at high frequencies, which is far more realistic than the previous results.
Figure 3.2: Dynamic rolling resistance using Pacejka’s model.
Figure 3.3: Influence of $\eta$ on dynamic rolling resistance.
3.3 Brush model and contact patch length [27]

The model developed by Zegelaar [27], using a brush model and incorporating the effect of a dynamic vertical load on the length of the contact patch (assumed to be elliptical) is compared with the experimental data available in [18]. The same data as previously is used. The values used for the model parameters in the simulations are detailed in Appendix A.4. The results are presented in Figure 3.4.

The static rolling resistance is well predicted by using an adequate value for $k_{cp}$. As opposed to the previous two models, the results do not show a resonance at higher frequencies, but instead the rolling resistance coefficient decreases with frequency before settling down to a constant value which is lower than the static rolling resistance coefficient. This is due to the exponential component in equation (2.17). It is worth noting that Zegelaar’s approach does not include the tyre vertical stiffness and therefore, the dynamic tyre stiffness derived from the flexible ring model is not used here. High frequency components in the vertical load generally mean that the road surface quality is deteriorating, which would indicate a higher rolling resistance than for low frequencies (at least in the region of the natural frequency of the axle hop vibration), since the fuel consumption on a rough road is higher than on a smooth surface. This observation, similarly to Segel and Lu’s adapted model, casts serious doubts on the validity of the approach.

3.4 Model comparison

In order to decide which model is best suited for rolling resistance calculations under dynamic vertical load, the predictions from the above three models are compared in Figure 3.5, together with the experimental results.

At low frequencies, all the models considered agree well with the experimental data, except Segel and Lu’s model combined with the dynamic tyre stiffness derived from the flexible ring model, as previously mentioned. As the frequency increases, the axle hop vibration is shown only when using the flexible ring model in conjunction with either Pacejka’s model or Segel and Lu’s model. However, only Pacejka’s model is able to predict the resonance due to the tyre-wheel rotation with respect to the contact patch. Zegelaar’s model matches the experimental data well at low frequencies, but gives unrealistic rolling resistance levels at high frequencies, as previously mentioned.

Overall, Pacejka’s model, combined with the dynamic tyre stiffness from the flexible ring model, is the only approach able to match the experimental results at low frequencies and predict the important features of tyre dynamics as well as the influence of the tyre-road contact. It also gives realistic values, even at the resonance peaks. Furthermore, it takes into account the fact that the vertical load consists of a static components and a dynamic one superimposed, which other models do not. Finally, the other two approaches yield values of rolling resistance at high frequencies which are lower than the static rolling resistance, which is not realistic, whereas Pacejka’s model predicts that the rolling resistance at high frequencies should return to the static value corresponding to steady rolling, which is far more plausible. There is some doubt as to the appropriate value for $\eta$, but even when using directly the value recommended by Pacejka, the results remain realistic.
(a) Over a large frequency range.

(b) At low frequencies.

Figure 3.4: Dynamic rolling resistance using Zegelaar’s model.
Figure 3.5: Comparison of rolling resistance predictions under dynamic vertical load using three different models: Experimental data (—×—), Pacejka’s model with flexible ring model (— — ), Pacejka’s model with spring and damper (— — ), Zegelaar’s model (····), Segel and Lu’s model with flexible ring model (—○—), Segel and Lu’s model with spring and damper (······).
Chapter 4

Conclusions

In this report, several existing tyre models are employed and modified to undertake rolling resistance calculations under dynamic vertical load for truck tyres. The models’ predictions are compared with the experimental data available from [18] and an assessment is carried out to determine which of the models considered is the best suited for this research project.

The flexible ring model developed by Gong [4] is detailed and suitable parameters are chosen for a truck tyre. Experimental modal analysis of the tyre showed that these parameters were adequate [12]. Two frequency response functions are derived from the model using appropriate boundary conditions. They represent the dynamic tyre stiffness for (a) cleat tests and (b) axle vibrations. The frequency response functions are found to model the tyre vibration properties well, but a discrepancy of 46% is found in the static tyre stiffness when compared to a standard spring and damper arrangement, where the stiffness was measured by static deflection of the tyre. The flexible ring model is integrated into a quarter-vehicle model for power consumption calculations [12]. The results show that for a typical value of suspension damping and for an average road profile, the suspension accounts for approximately 76% of the total energy loss with a constant speed of 80 km/h.

Three tyre models are considered for rolling resistance calculations:

- The first model was developed by Segel and Lu [21] and represents the viscoelastic properties of the tyre by radially distributed independent springs, each behaving as an hysteretic elastic element. A simple equation was derived to calculate the rolling resistance when going over a flat and even road. Their approach is combined with the dynamic tyre stiffness from the flexible ring model.

- The second model was developed by Pacejka [13] and considers the response of the axle forces to in-plane axle motions, road waviness and tyre non-uniformities. This fairly extensive derivation assumes small deviations from the undisturbed condition in order to perform a linear analysis, which yields the individual transfer functions, including the one of interest here: the frequency response to axle motions on flat and level ground. Pacejka’s model is combined the dynamic tyre stiffness from the flexible ring model to determine the rolling resistance coefficient.
• The last model considered here was developed by Zegelaar in [27]. He derived an analytical response to small variations of vertical load using a brush model. The effect of dynamic vertical load was modelled as variations in the contact patch length. In order to derive the rolling resistance coefficient, an elliptical contact patch is assumed so that the contact patch length can be expressed as a function of the vertical load.

The other tyre models considered [7, 26–28] were not found to be suitable for the purpose of this project.

The results show that all the models agree to a certain extent with the experimental data, obtained in the low frequency range (0–10 Hz). When considering a larger frequency range (0–50 Hz), the models exhibit very different characteristics. When the flexible ring model is used in conjunction with either Segel or Pacejka’s model, the resonance due to the axle hop vibration at approximately 25 Hz is shown. This is not case if a standard spring and damper arrangement is used instead of the dynamic tyre stiffness from the flexible ring model. However, it was noted that replacing the static tyre deflection by a dynamic one might not be a valid approach. Furthermore, only Pacejka’s model is able to predict the resonance due to the tyre-wheel rotation with respect to the contact patch at approximately 17 Hz and give realistic results throughout the frequency range considered. It is also the only approach taking into account both static and dynamic vertical load and giving realistic results at high frequencies, with the other two models giving a rolling resistance which is smaller than the static value. Consequently, Pacejka’s model, combined with the flexible ring model, was found to be appropriate for calculating rolling resistance under dynamic vertical load, based on the available experimental data. In order to make a more definite recommendation as to which model is the most suitable for rolling resistance calculations, more experimental data would be required at higher frequencies.
Appendix A

Tyre model parameters

A.1 Flexible ring model (Section 2.1)

<table>
<thead>
<tr>
<th>$\rho A$ [kg/m]</th>
<th>$b$ [m]</th>
<th>$h_0$ [m]</th>
<th>$R$ [m]</th>
<th>$A$ [m$^2$]</th>
<th>$I_r$ [kg.m$^2$]</th>
<th>$m$ [kg]</th>
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<tr>
<td>16.5</td>
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<td>0.02</td>
<td>0.518</td>
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<table>
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<tr>
<th>$EI$ [N.m$^2$]</th>
<th>$k_v$ [N/m$^2$]</th>
<th>$k_w$ [N/m$^2$]</th>
<th>$k_{Et}$ [N/m$^2$]</th>
<th>$k_{Gt}$ [N/m$^2$]</th>
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<tr>
<td>2.8</td>
<td>$3.8 \times 10^5$</td>
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<td>$7.65 \times 10^7$</td>
<td>$3.06 \times 10^7$</td>
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<tr>
<th>$k_t$ [N/m]</th>
<th>$c_t$ [N.s/m]</th>
<th>$p_0$ [bar]</th>
<th>$V_x$ [km/h]</th>
<th>$\Omega$ [rad/s]</th>
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<tbody>
<tr>
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<td>9.4</td>
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A.2 Radially distributed hysteretic springs (Section 2.2)

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<th>$R_0$ [m]</th>
<th>$R_e$ [m]</th>
<th>$a_{fr}$ [N/m-rad]</th>
<th>$k_w$ [N/m$^2$]</th>
<th>$k_{Et}$ [N/m$^2$]</th>
<th>$a_{sd}$ [N/m-rad]</th>
<th>$k_t$ [N/m]</th>
<th>$c_t$ [N.s/m]</th>
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</thead>
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<tr>
<td>0.536</td>
<td>0.525</td>
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<td>$2.82 \times 10^6$</td>
<td>$7.65 \times 10^7$</td>
<td>$1.32 \times 10^5$</td>
<td>$8.3 \times 10^5$</td>
<td>299.40</td>
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</tbody>
</table>

$a_{fr}$ refers to the value of $a$ used with the flexible ring model in equation (2.10) whereas $a_{sd}$ refers to the value of $a$ used with the simple spring and dashpot model in equation (2.11).
### A.3 Pacejka’s transient tyre model (Section 2.3)

<table>
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<tr>
<th>$\eta$</th>
<th>$A_r$</th>
<th>$r_f$ [m]</th>
<th>$r_e$ [m]</th>
<th>$I_w$ [kg.m$^2$]</th>
<th>$C_{F_h}$ [N]</th>
<th>$C_{Fx}$ [N/m]</th>
<th>$V$ [km/h]</th>
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<td>0.01</td>
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### A.4 Brush model and contact patch length (Section 2.4)

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<th>$R$ [m]</th>
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<th>$V_{cx0}$ [m/s]</th>
<th>$V_{cr0}$ [m/s]</th>
<th>$\Omega$ [rad/s]</th>
<th>$\zeta_{cx}$</th>
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</thead>
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<td>0.525</td>
<td>11.11</td>
<td>11.27</td>
<td>21.45</td>
<td>1.39 $\times$ 10$^{-2}$</td>
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<table>
<thead>
<tr>
<th>$F_{z0}$ [kN]</th>
<th>$q_{a1}$ [m/N$^{1/2}$]</th>
<th>$q_{a2}$ [m/N]</th>
<th>$a_0$ [m]</th>
<th>$k_{cp}$ [N/m$^2$]</th>
</tr>
</thead>
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<td>35</td>
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<td>3.13 $\times$ 10$^{-6}$</td>
<td>0.14</td>
<td>2 $\times$ 10$^5$</td>
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</tbody>
</table>
References


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